Time Series Regression and Forecasting

Juergen Meinecke

Roadmap

Time Series Terminology, Autocorrelation

Lags, First Differences, Growth Rates

Notation is now slightly different

Instead of an i-subscript, variables will have a t-subscript
(this is not a substantive change, just convention for time series)

The variable Y_t is the value of Y (for example real GDP) in period t
(for example year)

Data set: $\{Y_1, \dots, Y_T\}$ are T observations on the time series Y

We consider only consecutive, evenly-spaced observations
(for example, monthly, 1960 to 1999, no missing months)

Missing and unevenly spaced data do not pose a principal problem and only introduce technical complications which we are happy to ignore at this stage

Definition

The **first lag** of time series Y_t is Y_{t-1}.

The **j-th lag** of time series Y_t is Y_{t-j}.

Definition

The **first difference** of time series Y_t is $\Delta Y_t := Y_t - Y_{t-1}$.

Definition

The **first difference of the logarithm** of time series Y_t is $\Delta \ln(Y_t) := \ln(Y_t) - \ln(Y_{t-1})$.

With these definitions it is easy to determine the percentage change of a time series Y_t between the periods $t-1$ and t:
it is approximately $100 \cdot \Delta \ln(Y_t)$

Example: Quarterly CPI data for the US

I'm starting out with a time series on the price level in the US

Price level here is measured by the consumer price index (CPI)

The specific time series I'm using is labelled `CPIAUCSL`

It is the *Consumer Price Index for All Urban Consumers* provided by the Federal Reserve Bank of St. Louis (FRED)

Let's look at two recent measurements

- CPI in the fourth quarter of 2024 (2024:Q4) = 316.54
- CPI in the first quarter of 2025 (2025:Q1) = 319.49

Given this price level data, how do we back out inflation?

We study two approaches: exact and approximate

- CPI in the fourth quarter of 2024 (2024:Q4) = 316.54
- CPI in the first quarter of 2025 (2025:Q1) = 319.49
- Inflation via *exact* percentage change in CPI, 2023:Q4 to 2024:Q1

$$100 \cdot \left(\frac{319.49 - 316.54}{316.54} \right) = 0.932\%$$

- Inflation via logarithmic *approximation* instead:

$$100 \cdot (\ln(319.49) - \ln(316.54)) = 0.928\%$$

The two approaches give slightly different results

It is common to extrapolate up the quarter-to-quarter change to an annual rate

Quarter-to-quarter change *at an annual rate*

- Annualized inflation via *exact* percentage change in CPI

$$4 \cdot 100 \cdot \left(\frac{319.49 - 316.54}{316.54}\right) = 3.728\%$$

- Annualized inflation via logarithmic *approximation* instead:

$$4 \cdot 100 \cdot (\ln(319.49) - \ln(316.54)) = 3.711\%$$

Answers the question: if the current quarter inflation continued throughout the year, what would annual inflation be?

It’s a simple extrapolation really

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The correlation of a time series with its own lagged values is called autocorrelation or serial correlation

Definition

The j-th **autocovariance** of a time series Y_t is the covariance between Y_t and its j-th lag, Y_{t-j}: $\text{Cov}(Y_t, Y_{t-j})$.

The j-th **autocorrelation** of a time series Y_t is the correlation between Y_t and its j-th lag, Y_{t-j}:

$$\rho(j) := \frac{\text{Cov}(Y_t, Y_{t-j})}{\sqrt{\text{Var}\,(Y_t)\text{Var}\,(Y_{t-j})}}.$$

The sample autocorrelation is the *estimated* autocorrelation

Definition

The j-th **sample autocorrelation** of a time series Y_t is the correlation between Y_t and its j-th lag, Y_{t-j}:

$$\hat{\rho}(j) := \frac{\widehat{\text{Cov}}(Y_t, Y_{t-j})}{\widehat{\text{Var}}\,(Y_t)},$$

$$\text{with} \quad \widehat{\text{Cov}}(Y_t, Y_{t-j}) := \frac{1}{T}\sum_{t=j+1}^{T}(Y_t - \bar{Y}_{j+1,T})(Y_{t-j} - \bar{Y}_{1,T-j})$$

$$\bar{Y}_{p,q} := \frac{1}{T-j}\sum_{t=p}^{q} Y_t$$

$$\widehat{\text{Var}}\,(Y_t) := \frac{1}{T}\sum_{t=1}^{T}(Y_t - \bar{Y})^2$$

Two little comments:

Although we only compare $T - j$ pairs of the time series, the division is by T (this is conventional in time series analysis)

When computing the sample autocorrelation, we have implicitly assumed that

- variances are constant over time
- covariances are constant over time
  (only dependent on the lag length j)

This is justified by *stationarity* (which we will define next week)

Python example: quarterly CPI data for the US

Using the time series CPIAUCSL on quarterly CPI in the US,
I create the quarter-to-quarter inflation at an annualized rate

Python Code

```
> import pandas as pd
> import statsmodels.formula.api as smf
> import numpy as np
> # reading data from spreadsheet (downloaded from FRED):
> df = pd.read_csv('CPIAUCSL.csv')

> # creating quarterly index
> df['date'] = pd.to_datetime(df['DATE'], format='%Y-%m-%d')
> df.index = pd.DatetimeIndex(df.date, name='quarter').to_period('Q')

> # copy of CPI series with easy-to-access name:
> df['cpi'] = df.CPIAUCSL

> # taking logarithm of original series:
> df['logcpi'] = np.log(df.cpi)

> # creating annualised inflation via differences in logs:
> # (this is the 'first derivative' of 'cpi')
> df['infl'] = 400 * df.logcpi.diff()

> # creating quarter-on-quarter differences in inflation:
> # (this is the 'second derivative' of 'cpi')
> df['dinfl'] = df.infl.diff()

> df = df.drop(['DATE', 'CPIAUCSL'], axis=1)
```

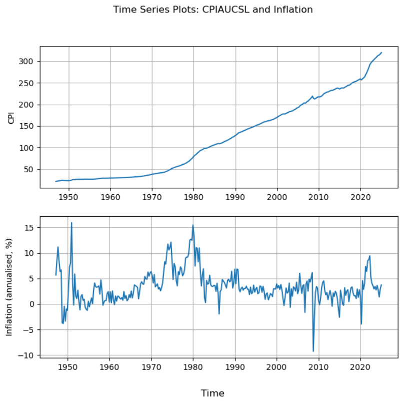
Let’s take a look at the time series

Python Code

```
> # looking at data: top two years
> print(df.head(8))
              date     cpi    logcpi       infl      dinfl
quarter
1947Q1  1947-01-01  21.700  3.077312        NaN        NaN
1947Q2  1947-04-01  22.010  3.091497   5.673854        NaN
1947Q3  1947-07-01  22.490  3.113071   8.629548   2.955694
1947Q4  1947-10-01  23.127  3.141001  11.172000   2.542452
1948Q1  1948-01-01  23.617  3.161967   8.386410  -2.785590
1948Q2  1948-04-01  23.993  3.177762   6.318132  -2.068278
1948Q3  1948-07-01  24.397  3.194460   6.679221   0.361089
1948Q4  1948-10-01  24.173  3.185236  -3.689546 -10.368768

> # looking at data: bottom two years
> print(df.tail(8))

              date      cpi    logcpi      infl     dinfl
quarter
2023Q2  2023-04-01  303.424  5.715131  2.953293 -0.640588
2023Q3  2023-07-01  306.042  5.723722  3.436472  0.483179
2023Q4  2023-10-01  308.158  5.730613  2.756116 -0.680356
2024Q1  2024-01-01  310.974  5.739709  3.638668  0.882551
2024Q2  2024-04-01  313.096  5.746510  2.720218 -0.918449
2024Q3  2024-07-01  314.183  5.749976  1.386306 -1.333912
2024Q4  2024-10-01  316.539  5.757446  2.988335  1.602029
2025Q1  2025-01-01  319.492  5.766732  3.714311  0.725976
```

Python Code

```
> fig, axs = plt.subplots(2, 1, figsize=(8,7))
> axs[0].plot(df.date, df.cpi)
> axs[0].set_ylabel('CPI')
> axs[1].plot(df.date, df.infl)
> axs[1].set_ylabel('Inflation (annualised, %)')
> fig.supxlabel('Time')
> fig.suptitle('Time Series Plots: CPIAUCSL and Inflation')
> plt.show()
```



Then I look at sample autocorrelations

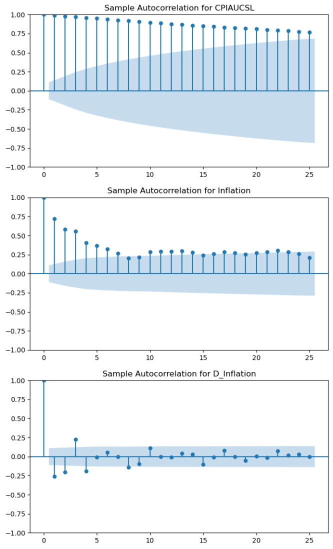
Python Code

```
> from matplotlib import pyplot as plt
> from statsmodels.graphics.tsaplots import plot_acf

> # creating a 'stacked' plot of 3 rows
> fig, axs = plt.subplots(3, 1, figsize = (8, 14))

> # stacking them
> plot_acf(df.cpi, ax=axs[0], title = 'Sample Autocorrelation for CPIAUCSL')
> plot_acf(df.infl, missing='drop', ax=axs[1], title = 'Sample Autocorrelation for Inflation')
> plot_acf(df.dinfl, missing='drop', ax=axs[2], title = 'Sample Autocorrelation for D_Inflation')
> plt.show()
```

which creates the following plot ...

Increasing degree of ‘differentiation’ reduces autocorrelation

Python Code Output



These sample autocorrelations show

- the original time series `CPIAUCSL` (price level as measured by cpi) is very highly serially or auto-correlated
- `infl` (the first derivative of `CPIAUCSL`) is still highly serially correlated
- `dinfl` (the first derivative if `infl` and second derivative of `CPIAUCSL`) is not serially correlated anymore

Please bear this in mind, as it will have important ramifications when we want to run auto-regressions using price level or inflation data

Detecting serial correlation by visual inspection is tricky:
both series are highly auto-correlated, yet only obvious for CPI

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Roadmap

Autoregressive Models and Forecasting

The First Order Autoregressive (AR(1)) Model

A natural starting point for a forecasting model is to use past values of Y (that is, $Y_{t-1}, Y_{t-2}, \ldots$) to forecast Y_t

An autoregression is a regression model in which Y_t is regressed against its own lagged values

The number of lags used as regressors is called the *order* of the autoregression

In a first order autoregression, Y_t is regressed against Y_{t-1}

In a p-th order autoregression, Y_t is regressed against $Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p}$

The population AR(1) model is

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + u_t$$

The coefficient β_1 does NOT have a causal interpretation

If $\beta_1 = 0$ then Y_{t-1} is not useful for forecasting Y_t

The AR(1) model is estimated by OLS regression of Y_t on Y_{t-1}

Testing $\beta_1 = 0$ versus $\beta_1 \neq 0$ provides a test of the hypothesis that Y_{t-1} is not useful for forecasting Y_t

Python Code

```
> # creating lagged inflation
> # (will be used as explanatory variable in AR(1) estimation)
> df['l1infl'] = df.infl.shift(1)

> # looking at data: top two years
> print(df[['cpi', 'infl', 'l1infl']].head(8))
            cpi       infl     l1infl
quarter
1947Q1   21.700        NaN        NaN
1947Q2   22.010   5.673854        NaN
1947Q3   22.490   8.629548   5.673854
1947Q4   23.127  11.172000   8.629548
1948Q1   23.617   8.386410  11.172000
1948Q2   23.993   6.318132   8.386410
1948Q3   24.397   6.679221   6.318132
1948Q4   24.173  -3.689546   6.679221


> # looking at data: bottom two years
> print(df[['cpi', 'infl', 'l1infl']].tail(8))
             cpi      infl    l1infl
quarter
2023Q2   303.424  2.953293  3.593881
2023Q3   306.042  3.436472  2.953293
2023Q4   308.158  2.756116  3.436472
2024Q1   310.974  3.638668  2.756116
2024Q2   313.096  2.720218  3.638668
2024Q3   314.183  1.386306  2.720218
2024Q4   316.539  2.988335  1.386306
2025Q1   319.492  3.714311  2.988335
```

Here I'm running an AR(1) estimation for `infl`

Python Code (output edited)

```
> # first order autoregression:
> ar1 = smf.ols('infl ~ l1infl', data=df, missing='drop').fit(use_t=False)
> print(ar1.summary())

OLS Regression Results
==============================================================================
Dep. Variable:                   infl   R-squared:                       0.522
Model:                            OLS   Adj. R-squared:                  0.520
Method:                 Least Squares   F-statistic:                     337.1
No. Observations:                 311
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept      0.9529      0.184      5.176      0.000       0.592       1.314
l1infl         0.7217      0.039     18.360      0.000       0.645       0.799
==============================================================================
```

Notice: We don't need to use heteroskedasticity-robust standard errors because we are not really interested in statistical inference, instead we want to use the coefficient estimates to produce forecasts

Forecasting

Our main objective when estimating autoregressions is to produce *forecasts*

We are not interested in causal effects

As a consequence, we are not usually interested in the coefficient estimates of AR models

We only use the coefficient estimates to create a forecast for the dependent variable

External validity is paramount: the model estimated using historical data must hold into the (near) future

But what do I mean by *forecast*?

Notation

- For an AR(1) model:

$$Y_{T+1|T} = \beta_0 + \beta_1 Y_T$$
$$\hat{Y}_{T+1|T} = \hat{\beta}_0 + \hat{\beta}_1 Y_T$$

- $Y_{T+1|T}$: forecast of Y_{T+1} based on $Y_T, Y_{T-1}, \dots$ using the population coefficients (typically unknown)
- $\hat{Y}_{T+1|T}$: forecast of Y_{T+1} based on $Y_T, Y_{T-1}, \dots$ using the estimated coefficients
- Forecast errors are defined by $Y_{T+1} - \hat{Y}_{T+1|T}$

Do not confuse predicted values with forecasts

- *Predicted values* are "in-sample"
- *Forecasts* are "out-of-sample" (looking into the future)

Let me explain the difference between predicted values and forecasts

Earlier we estimated the following AR(1) model for inflation:

$$\widehat{\texttt{infl}}_t = 0.9529 + 0.7217 \cdot \texttt{infl}_{t-1}$$

We used data from 1947:Q1–2025:Q1 for the estimation

This means:

- $\widehat{\texttt{infl}}_{2025:Q1}$ is a predicted value
- $\widehat{\texttt{infl}}_{2025:Q2|2025:Q1}$ is a forecast

Let's calculate both

These are simple common sense calculations

Calculation for the predicted value

In the data we observe $\texttt{infl}_{2024:Q4} = 2.9883$

Resulting in the predicted value

$$\widehat{\texttt{infl}}_{2025:Q1} = 0.9529 + 0.7217 \cdot 2.9883 = 3.1097$$

In my data set I do observe $\texttt{infl}_{2025:Q1} = 3.7143$
therefore $\texttt{infl}_{2025:Q1} - \widehat{\texttt{infl}}_{2025:Q1}$ is the *residual* for Q1 2025

Calculation for the forecast

In the data we observe $\texttt{infl}_{2025:Q1} = 3.7143$

Resulting in the forecast values

$$\widehat{\texttt{infl}}_{2025:Q2|2025:Q1} = 0.9529 + 0.7217 \cdot 3.7143 = 3.6337$$

I could wait until July when $\texttt{infl}_{2025:Q2}$ is released and calculate the *forecast error* $\texttt{infl}_{2025:Q2} - \widehat{\texttt{infl}}_{2025:Q2|2025:Q1}$

Easy to produce predicted values and forecasts in `Python`

Just use the post-regression `predict` function

It will produce a predicted value when in-sample

It will produce a forecast value when out-of-sample

Python Code

```
> # Prediction for 2025:Q1, and forecast for 2025:Q2
> newdata = {'l1infl' : [df.infl[-2], df.infl[-1]]}
> ar1.predict(newdata)

0    3.109693
1    3.633662
```

Time Series Regression and Forecasting

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Roadmap

Autoregressive Models and Forecasting

- The p-th Order Autoregressive (AR(p)) Model

The population AR(p) model is

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \ldots + \beta_p Y_{t-p} + u_t$$

The coefficients do NOT have a causal interpretation

To test hypothesis that $Y_{t-2}, \ldots, T_{t-p}$ do not add value over and above Y_{t-1}, use an F-test

We will look at choosing p using a suitable information criterion

Here I'm preparing an AR(4) estimation for `infl`

Python Code

```
> # creating more lags for inflation
> df['l2infl'] = df.infl.shift(2)
> df['l3infl'] = df.infl.shift(3)
> df['l4infl'] = df.infl.shift(4)

># looking at data: top two years
> print(df[['cpi', 'infl', 'l1infl', 'l2infl', 'l3infl', 'l4infl']].head(8))
            cpi       infl     l1infl     l2infl     l3infl     l4infl
quarter
1947Q1   21.700        NaN        NaN        NaN        NaN        NaN
1947Q2   22.010   5.673854        NaN        NaN        NaN        NaN
1947Q3   22.490   8.629548   5.673854        NaN        NaN        NaN
1947Q4   23.127  11.172000   8.629548   5.673854        NaN        NaN
1948Q1   23.617   8.386410  11.172000   8.629548   5.673854        NaN
1948Q2   23.993   6.318132   8.386410  11.172000   8.629548   5.673854
1948Q3   24.397   6.679221   6.318132   8.386410  11.172000   8.629548
1948Q4   24.173  -3.689546   6.679221   6.318132   8.386410  11.172000

> # looking at data: bottom two years
> print(df[['cpi', 'infl', 'l1infl', 'l2infl', 'l3infl', 'l4infl']].tail(8))
             cpi      infl    l1infl    l2infl    l3infl    l4infl
quarter
2023Q2   303.424  2.953293  3.593881  4.026929  5.255211  9.445084
2023Q3   306.042  3.436472  2.953293  3.593881  4.026929  5.255211
2023Q4   308.158  2.756116  3.436472  2.953293  3.593881  4.026929
2024Q1   310.974  3.638668  2.756116  3.436472  2.953293  3.593881
2024Q2   313.096  2.720218  3.638668  2.756116  3.436472  2.953293
2024Q3   314.183  1.386306  2.720218  3.638668  2.756116  3.436472
2024Q4   316.539  2.988335  1.386306  2.720218  3.638668  2.756116
2025Q1   319.492  3.714311  2.988335  1.386306  2.720218  3.638668
```

Here I'm running an AR(4) estimation for `infl`

Python Code (output edited)

```
> # fourth order autoregression:
> ar4 = smf.ols('infl ~ l1infl + l2infl + l3infl + l4infl',
               data=df, missing='drop').fit(use_t=False)
> print(ar4.summary())

OLS Regression Results
==============================================================================
Dep. Variable:                   infl   R-squared:                       0.559
Model:                            OLS   Adj. R-squared:                  0.553
Method:                 Least Squares   F-statistic:                     95.92
No. Observations:                 308
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept      0.7555      0.192      3.929      0.000       0.379       1.132
l1infl         0.6176      0.057     10.905      0.000       0.507       0.729
l2infl         0.0073      0.064      0.114      0.909      -0.118       0.133
l3infl         0.3138      0.064      4.895      0.000       0.188       0.439
l4infl        -0.1672      0.056     -2.983      0.003      -0.277      -0.057
==============================================================================
```

Again producing prediction and forecast

We estimated the following AR(4) model for inflation:

$$\widehat{\texttt{infl}}_t = 0.7555 + 0.6176 \cdot \texttt{infl}_{t-1} + 0.0073 \cdot \texttt{infl}_{t-2} + 0.3138 \cdot \texttt{infl}_{t-3} - 0.1672 \cdot \texttt{infl}_{t-4}$$

In the data we observe

Python Code

```
> df.infl.tail(5)
quarter
2024Q1    3.638668
2024Q2    2.720218
2024Q3    1.386306
2024Q4    2.988335
2025Q1    3.714311
```

$$\widehat{\texttt{infl}}_{2025:Q1} = 0.7555 + 0.6176 \cdot (2.9883) + 0.0073 \cdot (1.3863) + 0.3138 \cdot (2.7202) - 0.1672 \cdot (3.6387) = \mathbf{2.8563}$$

$$\widehat{\texttt{infl}}_{2025:Q2|2025:Q1} = 0.7555 + 0.6176 \cdot (3.7143) + 0.0073 \cdot (2.9883) + 0.3138 \cdot (1.3863) - 0.1672 \cdot (2.7202) = \mathbf{3.0514}$$

Forecasting

Still easy to produce predicted values and forecasts in `Python`

Again use the post-regression `predict` function

It will produce a predicted value when in-sample

It will produce a forecast value when out-of-sample

Python Code

```
> # Prediction for 2025:Q1, and forecast for 2025:Q2
> newdata = {'l1infl' : [df.infl[-2], df.infl[-1]],
             'l2infl' : [df.infl[-3], df.infl[-2]],
             'l3infl' : [df.infl[-4], df.infl[-3]],
             'l4infl' : [df.infl[-5], df.infl[-4]]}
> ar4.predict(newdata)

0    2.856292
1    3.051373
```