Advanced Econometrics I
EMET4314/8014
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Juergen Meinecke
Research School of Economics
ANU

**Problem Set 3**
(due: Monday week 4, 11:00am)

**Submission Instructions:** Same as last week.

# Exercises

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

1. Let $Z$ be a random variable with $\mathrm{E}Z^2 < \infty$. Prove that $Z_N \overset{\mathrm{d}}{\to} Z$ implies $Z_N = \mathrm{O}_p(1)$.

2. The pdf of a normal distribution is $f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right)$, for $-\infty < y < \infty$.

   (a) Derive the moment generating function of a normally distributed random variable. Denote it by $M_Y(t;(\mu,\sigma))$.

   (b) Take the first two derivatives of $M_Y(t;(\mu,\sigma))$ and evaluate them at zero.

   (c) Evaluate the mgf for the standard normal case: $M_Y(t;(0,1))$.
   (This proves a Lemma from the week 3 lecture notes.)

3. Let $Y = X\beta^* + u$ with $\dim X = N \times K$ and the usual definitions. Define the *projection matrix* $P_X := X(X'X)^{-1}X'$ and the *residual maker matrix* $M_X := I_N - P_X$. Show that:

   (i) $P_X Y = \hat{Y}$ (hence the name *projection matrix*)

   (ii) $M_X Y = \hat{u}$ (hence the name *residual maker matrix*)

   (iii) $M_X u = \hat{u}$

   (iv) Symmetry: $P_X = P_X'$ and $M_X = M_X'$

   (v) Idempotency: $P_X P_X = P_X$ and $M_X M_X = M_X$

   (vi) tr $P_X = K$ and tr $M_X = N - K$

4. Use a derivation similar to lecture notes 3 to show that $\sum_{i=1}^N \hat{u}_i^2/(N-K)$ is an unbiased estimator for $\sigma_u^2$.

5. Consider the asymptotic distribution of $\sqrt{N}(\hat{\beta}^{\text{OLS}} - \beta^*)$ under the assumption of *homoskedasticity*, that is: $\text{E}(u_i^2|X_i) = \sigma_u^2$ where $\sigma_u^2 \in \mathbb{R}$. Note, as usual, $\beta^* = \text{E}(X_i X_i')^{-1}\text{E}(X_i Y_i)$.

   (a) Derive the asymptotic distribution of $\sqrt{N}(\hat{\beta}^{\text{OLS}} - \beta^*)$ under homoskedasticity. Justify each step!

   (b) Suggest a consistent estimator for the asymptotic variance of $\sqrt{N}(\hat{\beta}^{\text{OLS}} - \beta^*)$ under homoskedasticity.

   (c) Prove that your estimator from part (b) is consistent. In your proof, make use of the $\text{o}_p(1)$ and $\text{O}_p(1)$ notation. Justify each step!