

### Problem Set 10

(due: Monday week 12, 11:00am)

**Submission Instructions:** Same as last week.

## Exercise

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

The *normal regression model* is

$$Y_i = X_i' \beta + e_i, \quad \text{where } e_i | X_i \sim N(0, \sigma_e^2),$$

so the errors have an *exact* normal distribution. The unknown parameters are  $\beta \in \mathbb{R}^K$  and  $\sigma_e^2$ .

Notice that the above normal regression model can be regarded, equivalently, as a statement about the density of  $Y_i$  given  $X_i$ . That conditional density is

$$f_{Y|X}(y|x, \beta, \sigma_e^2) = \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{1}{2\sigma_e^2}(y - x'\beta)^2\right).$$

- (i) Derive the likelihood function  $L(\beta, \sigma_e^2)$ .
- (ii) Derive the log likelihood function  $\ln L(\beta, \sigma_e^2)$ .
- (iii) Derive  $(\hat{\beta}^{\text{ML}}, \hat{\sigma}_e^{2,\text{ML}}) := \underset{\beta \in \mathbb{R}^K, \sigma_e^2 > 0}{\operatorname{argmax}} \ln L(\beta, \sigma_e^2)$ . Does  $\hat{\beta}^{\text{ML}}$  look familiar?
- (iv) Derive the Hessian matrix as the derivative of the score

$$S(y | x, \beta, \sigma_e^2) := \begin{pmatrix} \frac{\partial \ln f_{Y|X}}{\partial \beta} (y|x, \beta, \sigma_e^2) \\ \frac{\partial \ln f_{Y|X}}{\partial \sigma_e^2} (y|x, \beta, \sigma_e^2) \end{pmatrix}.$$

The Hessian is obtained according to:

$$H(y | x, \beta, \sigma_e^2) := \begin{pmatrix} \frac{\partial^2 \ln f_{Y|X}}{\partial \beta \partial \beta'} (y|x, \beta, \sigma_e^2) & \frac{\partial^2 \ln f_{Y|X}}{\partial \beta \partial \sigma_e^2} (y|x, \beta, \sigma_e^2) \\ \frac{\partial^2 \ln f_{Y|X}}{\partial \sigma_e^2 \partial \beta'} (y|x, \beta, \sigma_e^2) & \frac{\partial^2 \ln f_{Y|X}}{\partial (\sigma_e^2)^2} (y|x, \beta, \sigma_e^2) \end{pmatrix}.$$

- (v) Derive the Fisher information  $I(\beta, \sigma_e^2)$  by calculating  $-\mathbb{E}(H(Y_i | X_i, \beta, \sigma_e^2))$ .
- (vi) State the asymptotic distribution for  $\sqrt{N}(\hat{\beta}^{\text{ML}} - \beta)$  and  $\sqrt{N}(\hat{\sigma}_e^{2,\text{ML}} - \sigma_e^2)$ .