Advanced Econometrics I
EMET4314/8014
Semester 1, 2026

Juergen Meinecke
Research School of Economics
ANU

Problem Set 5
(NOT due: Monday week 6, 11:00am)
(due: Monday week 7, 11:00am)

Submission Instructions: Same as last week.

Exercises

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

1. Consider the scalar model $Y_i = \beta_1 X_{i1} + \beta_2 X_{i2} + e_i$ where both variables are endogenous, that is, $\mathrm{E}(e_i X_{i1}) \neq 0$ and $\mathrm{E}(e_i X_{i2}) \neq 0$. You have available one instrumental variable Z_i (also a scalar) such that $\mathrm{E}(e_i Z_i) = 0$ and $\mathrm{E}(X_{i1} Z_i) \neq 0$ and $\mathrm{E}(X_{i2} Z_i) \neq 0$.

   Consider the two reduced form regressions

   $$\begin{aligned} X_{i1} &= \pi Z_i + v_i \\ X_{i2} &= \gamma Z_i + w_i, \end{aligned}$$

   where $\mathrm{E}(v_i X_{i1}) = 0$ and $\mathrm{E}(w_i X_{i2}) = 0$.

   Show that a regression of Y_i on Z_i cannot separately identify β_1 and β_2, but instead only $\pi\beta_1 + \gamma\beta_2$.

   Contrast this with the case in which you have two instruments:

   $$\begin{aligned} X_{i1} &= \pi_1 Z_{i1} + \pi_2 Z_{i2} + v_i \\ X_{i2} &= \gamma_1 Z_{i1} + \gamma_2 Z_{i2} + w_i. \end{aligned} \tag{1}$$

   Plugging into the structural equation results in

   $$Y_i = (\beta_1\pi_1 + \beta_2\gamma_1) Z_{i1} + (\beta_1\pi_2 + \beta_2\gamma_2) Z_{i2} + (e_i + \beta_1 v_i + \beta_2 w_i). \tag{2}$$

   OLS estimation of the two reduced form regressions (1) will produce consistent estimates for π_1, π_2, γ_1 and γ_2. Likewise, OLS estimation of the transformed structural equation (2) will produce consistent estimates for $(\beta_1\pi_1 + \beta_2\gamma_1)$ and $(\beta_1\pi_2 + \beta_2\gamma_2)$. You are now able to back out estimates for β_1 and β_2, because you essentially are facing a problem of solving two equations for two unknowns. (When you only have one instrumental variable, then you are solving one equation for two unknowns.)

   Note: this exercise illustrates why the number of instrumental variables must be at least as large as the number of endogenous regressors.

2. Remember the following parameter from the week 5 lecture:

$$\pi = \mathrm{E}(Z_i Z_i')^{-1}\mathrm{E}(Z_i X_i'),$$

   where $X_i = (X_{i1}', X_{i2}')'$ and $Z_i = (X_{i1}', Z_{i2}')'$. I claimed that π is an upper triangular block matrix. Show this!

   Hint: Define $A := \mathrm{E}(Z_i Z_i')$ and $B := \mathrm{E}(Z_i X_i')$, so that $\pi = A^{-1}B$. Multiply both sides by A to obtain $A\pi = B$. Now compare the entries of A and B to make the case that π must be upper triangular.

3. Let $Y_i = X_i\beta + e_i$ where X_i is a scalar random variable and $\mathrm{E}(e_i|X_i) = 0$. You observe $(\widetilde{X}_i, Y_i)$ with $\widetilde{X}_i := X_i + r_i$ where r_i is a random error. Derive the probability limit of the OLS estimator in the regression of Y_i on $\widetilde{X}_i$. For simplicity, assume that $\mathrm{E}X_i = \mathrm{E}r_i = 0$.

   What is the probability limit equal to when the variance of r_i is zero? When the variance of r_i is very large? What is the intuition here?

   Hint: Your probability limit should have the form $\beta(1 - \textit{stuff})$, where *stuff* depends only on the population variances of r_i and X_i.

   Note: This phenomenon is usually referred to as *measurement error bias* or *attenuation bias*. It is not good to use the word *bias* here though. After all, we are not studying the expected value of the OLS estimator, instead we are studying its probability limit and show that it does not equal β. Biasedness and inconsistency are not the same thing.

4. Prove that $\hat{\beta}^{\mathrm{IV}}$ is consistent. In your derivation, employ the $\mathrm{o}_p(1)$ and $\mathrm{O}_p(1)$ notation!

5. Derive the asymptotic distribution of $\sqrt{N}(\hat{\beta}^{\mathrm{IV}} - \beta)$. In your derivation, employ the $\mathrm{o}_p(1)$ and $\mathrm{O}_p(1)$ notation!

6. Read the paper "*A Practical Guide to Weak Instruments*" by Keane and Neal, published 2024 in the Annual Review of Economics.

   Try to read the whole paper and make sense of as much as you possible. I don't expect you to understand everything, but hopefully you will be able to follow the main points. The objective is to prepare you for the weeks 8 and 9 lectures in which we will discuss the paper in more detail.