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**Assignment 8**
(due: Tuesday week 9, 11:00am)

**Submission Instructions:** Same as last week.

# Exercises

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

Consider the *normal regression model*

$$Y_i = \beta X_i + e_i,$$

where $X_i$ and $Y_i$ are scalars and $e_i|X_i \sim \mathrm{N}(0, \sigma_e^2)$. This model makes the strong assumption that the conditional error distribution is exactly normal.

(i) Derive the conditional distribution of the OLS estimator for $\beta$, that is come up with a statement like

$$\hat{\beta}^{\text{OLS}}|(X_1, \dots, X_N) \sim \dots,$$

where you determine the "…" part.

Notice that you are deriving the *exact* distribution here, not the approximate one. You do not need to use asymptotic theory because of the normality assumption imposed on the error distribution.

Read the primer on hypothesis testing on the next page to answer the following exercises.

(ii) Propose a sensible test of $H_0 : \beta = 0$ with an exact size of 5%.
(Please assume that you know the value of $\sigma_e^2$.)

Aside: It is easy to propose a non-sensible test: draw a number randomly from $\{1, 2, 3, \dots, 20\}$ and reject the null if that number equals $8$. If the null is true you will reject it with a probability of $1/20 = 5$%. Unfortunately, the power of that test is also $5$%, which is sad.

(iii) Derive the power function $\pi(\beta)$ for the normal regression model. Let $H_0 : \beta = 0$.

Sketch $\pi(\beta)$ either by hand or computer. This does not need to be very precise.

# Primer on Hypothesis Testing

Let me illustrate everything with the simple univariate model $Y_i \sim \text{N}(\beta, 1)$.
You do not know $\beta$, the value of the coefficient that generates $Y_i$. But you have available a random sample $Y_i$, $i = 1, \ldots, N$ and you would like to know if $\beta \stackrel{?}{=} 0$.
You formulate that question by way of hypotheses:

$$H_0 : \beta = 0$$
$$H_1 : \beta \neq 0$$

You conduct a statistical test:

**Definition.** *Let $T := g(Y_1, \ldots, Y_N)$ be some statistic. A **statistical test** is a decision rule about a null hypothesis that compares $T$ to some critical value $c$ and decides to*

*reject* $H_0$ *if* $|T| > c$
*not reject* $H_0$ *if* $|T| \leq c$

A simple and silly example would be a decision rule that uses $T = \bar{Y}$ and decides $\beta \neq 0$ (reject $H_0$) whenever $|T| > 42$.
A more conventional example is the decision rule that uses $T = \bar{Y}/\text{se}(\bar{Y})$ and decides $\beta \neq 0$ (reject $H_0$) whenever $|T| > 1.96$. This is the simpel $t$-test.
There are two types of errors that you can make with a statistical test:

- type I error: to reject $H_0$ when it is true
- type II error: not to reject $H_0$ when it is false

Ideally we want to minimize the probability of these two errors. Usually, however, the two goals of minimizing these errors stand in contradiction to each other, and the conventional approach is to fix the type I error at some low level (it is seen as the worse of the two errors) and then cross your fingers (and close your eyes) for the type II error.
The power function is a convenient way to graph the type II error as a function of the true value of $\beta$. Because the $Y_i$ are determined by the value of $\beta$, your statistic $T$ also is determined by the value of $\beta$. Denote the distribution of $T$ under the particular value of $\beta$ by $\mathbb{P}_\beta(T)$. If you know $\mathbb{P}_\beta(T)$ then you can ask yourself questions such as: *what is the probability of rejecting $H_0$ for particular values of $\beta$?* You would answer this question by deriving the distribution $\mathbb{P}_\beta(|T| > c)$. This is the idea behind the *power function*:

$$\pi(\beta) := \mathbb{P}_\beta(|T| > c).$$

The power function answers this question: For a given value of $\beta$, what is the probability that I reject my null hypothesis? When $H_0 : \beta = 0$ then $\pi(0)$ will give you the probability of a type I error (size), in all other cases it will give you 1 minus the probability of a type II error.