Advanced Econometrics I
EMET4314/8014
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Assignment 11
(due: Tuesday week 12, 11:00am)

Submission Instructions: Same as last week.

Exercise

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

The *normal regression model* is

$$Y_i = X_i'\beta + e_i, \qquad \text{where } e_i|X_i \sim \mathrm{N}(0, \sigma_e^2),$$

so the errors have an *exact* normal distribution. The unknown parameters are $\beta \in \mathbb{R}^K$ and σ_e^2.
Notice that the above normal regression model can be regarded, equivalently, as a statement about the density of Y_i given X_i. That conditional density is

$$f_{Y|X}(y|x, \beta, \sigma_e^2) = \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{1}{2\sigma_e^2}(y - x'\beta)^2\right).$$

You have available a random sample (X_i, Y_i), where the Y_i are iid with pdf $f_{Y|X}(y|x)$.

(i) Derive the likelihood function $L(\beta, \sigma_e^2)$.

(ii) Derive the log likelihood function $\ln L(\beta, \sigma_e^2)$.

(iii) Derive $\left(\hat{\beta}^{\mathrm{ML}}, \hat{\sigma}_e^{2,\mathrm{ML}}\right) := \underset{b \in \mathbb{R}^K, s^2 > 0}{\operatorname{argmax}} \ln L(b, s^2)$. Does $\hat{\beta}^{\mathrm{ML}}$ look familiar?

(iv) Derive the Hessian matrix as the derivative of the score

$$S(y|x, \beta, \sigma_e^2) := \begin{pmatrix} \frac{\partial \ln f_{Y|X}}{\partial \beta}(y|x, \beta, \sigma_e^2) \\ \frac{\partial \ln f_{Y|X}}{\partial \sigma_e^2}(y|x, \beta, \sigma_e^2). \end{pmatrix}$$

Your Hessian should have the following structure:

$$H(x, y) = \begin{pmatrix} \frac{\partial^2 \ln f_y}{\partial \beta \partial \beta'}(y|x, \beta, \sigma_e^2) & \frac{\partial^2 \ln f_y}{\partial \beta \partial \sigma_e^2}(y|x, \beta, \sigma_e^2) \\ \frac{\partial^2 \ln f_y}{\partial \sigma_e^2 \partial \beta'}(y|x, \beta, \sigma_e^2) & \frac{\partial^2 \ln f_y}{\partial (\sigma_e^2)^2}(y|x, \beta, \sigma_e^2) \end{pmatrix}$$

(v) Derive the Fisher information $I(\beta, \sigma_e^2)$ by calculating $-\mathrm{E}(H(X_i, Y_i))$.

(vi) State the asymptotic distribution for $\sqrt{N}\left(\hat{\beta}^{\mathrm{ML}} - \beta\right)$ and $\sqrt{N}\left(\hat{\sigma}_e^{2,\mathrm{ML}} - \sigma_e^2\right)$.