Assignment 4
(due: Tuesday week 5, 11:00am)

Submission Instructions: Same as last week.

The solutions will be discussed in the Friday workshop during week 5. Please let me know which exercises I should focus on.

Exercises

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

1. Prove that the OLS estimator $\hat{\beta}^{\text{OLS}}$ for β in the linear regression model is consistent.

2. The linear regression model in matrix format is $Y = X\beta + e$, with the usual definitions. Let $\text{E}(e|X) = 0$ and

$$
\text{E}(ee'|X) = \sigma^2\Gamma = \sigma^2
\begin{bmatrix}
\gamma_1 & 0 & 0 & \cdots & 0 \\
0 & \gamma_2 & 0 & \cdots & 0 \\
\vdots & & & \ddots & 0 \\
0 & 0 & 0 & \cdots & \gamma_N
\end{bmatrix}
=
\begin{bmatrix}
\sigma_1^2 & 0 & 0 & \cdots & 0 \\
0 & \sigma_2^2 & 0 & \cdots & 0 \\
\vdots & & & \ddots & 0 \\
0 & 0 & 0 & \cdots & \sigma_N^2
\end{bmatrix}
= \Sigma.
$$

Notice that as a covariance matrix, Σ is symmetric and nonnegative definite.

   (i) Derive $\text{Var}(\hat{\beta}^{\text{OLS}}|X)$.

   (ii) Let $\tilde{\beta} := CY$ be any other linear unbiased estimator where C' is an $N \times K$ function of X. Prove $\text{Var}(\tilde{\beta}|X) \geq (X'\Sigma^{-1}X)^{-1}$.

   (iii) An oracle tells you Γ. Let $\tilde{Y} := \Gamma^{-1/2}Y$ and $\tilde{X} := \Gamma^{-1/2}X$. Define the *generalized least squares (GLS)* estimator $\hat{\beta}_{GLS} := (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{Y}$. This defines the GLS estimator as the OLS estimator of $\tilde{Y}$ on $\tilde{X}$.

   Derive $\text{Var}(\hat{\beta}_{GLS}|X)$. How does it compare to $\text{Var}(\tilde{\beta}|X)$ from part (ii)?