Advanced Econometrics I
EMET4314/8014
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Assignment 1
(due by: Thursday week 2, 11:00am)

Submission Instructions

Online submission per Wattle file upload only. Only pdf-files accepted. Your solution must be in your own handwriting. You can write on paper and use a suitable app on your phone to scan and upload a pdf-file, or you can use a stylus-type pen on your tablet-type device.
The solutions will be discussed in the Friday workshop during week 2. Please let me know which exercises I should focus on.

Exercises

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

1. Consider the space $Z = (0, 1]$ equipped with the metric $d(x, y) = |x - y|$. Consider the following sequence in Z: $x_n = 1/n, n = 1, 2, \ldots$. Is it a Cauchy sequence that converges in Z?

2. Let X, Y be elements from a Hilbert space. Prove:

   (i) Cauchy-Schwarz inequality: $|\langle X, Y\rangle| \leq \|X\| \cdot \|Y\|$

   (ii) Triangle inequality: $\|X + Y\| \leq \|X\| + \|Y\|$

3. Prove: If $\mathrm{E}\left(X^2\right) < \infty$ and $\mathrm{E}\left(Y^2\right) < \infty$, then

   - $|\mathrm{E}X| < \infty$ and $|\mathrm{E}Y| < \infty$;
   - $|\mathrm{E}(XY)| < \infty$;
   - $|\mathrm{Cov}(X, Y)| < \infty$.

   This is useful: to guarantee existence of covariances, we only need finite second moments. That is why we define L_2 to be the space of random variables with finite second moments.

   Related useful fact (for your enjoyment, no need to prove):
   $\mathrm{E}\left(|Y|^p\right) < \infty$ implies $\mathrm{E}\left(|Y|^q\right) < \infty$ for $1 \leq q \leq p$
   (by Liapunov's inequality).

4. Prove: $\text{Cov}(X, Y) = \text{Cov}(X, \text{E}(Y|X))$

5. Prove: if $X \in \{0, 1\}$ then $\frac{\text{Cov}(X,Y)}{\text{Var}X} = \text{E}(Y|X = 1) - \text{E}(Y|X = 0)$.

6. Consider the space L_2, as defined in the lecture. Let $X, Y \in L_2$. Prove that $\text{E}(XY)$ is an inner product.

7. Let $X_2, X_3, Y \in L_2$. Find the projection of Y on $\text{sp}\,(X_2, X_3)$. (What I'm trying to say here is that you are NOT including the constant in the span.)

   Use the following Gram-Schmidt orthogonalization procedure to construct an orthonormal set:

   Lemma 1 (Gram-Schmidt). *Let $V_1, V_2, V_3, \ldots$ be a linearly independent sequence in an inner product space. Set $U_1 = V_1/||V_1||$, and define recursively:*

   $$U_k = \frac{V_k - \sum_{i=1}^{k-1} \langle V_k, U_i \rangle U_i}{\left\| V_k - \sum_{i=1}^{k-1} \langle V_k, U_i \rangle U_i \right\|}, \qquad \textit{for } k = 2, 3, \ldots$$

   Then $U_1, U_2, U_3, \ldots$ is an orthonormal sequence with sp$(U_1, U_2, \ldots, U_k) =$ sp$(V_1, V_2, \ldots, V_k)$.

8. Let $X_1, \ldots, X_K, Y \in L_2$. Use calculus to derive the following:

   $$\tilde{\beta} := \underset{b \in \mathbb{R}^K}{\text{argmin}}\, \text{E}\left((Y - X'b)^2 \right),$$

   where $X := (X_1, \ldots, X_K)'$ so that $\dim X = \dim b = K \times 1$.

   This demonstrates that $\mathbb{P}_{\text{sp}(X_1,\ldots,X_K)}Y$ can also be obtained by "traditional" methods.

9. Go to my Github website, click `Computer Labs` and follow the four steps under `Preparing your computer`.

   The objective of this exercise is to prepare you for the week 2 computer lab. We don't expect you to know any Julia or be familiar with VS Code. Using VS Code and learning Julia will be covered in the computer labs. But we want you to be ready so that we can focus on the econometric coding as soon as possible!

10. Notice: Assignment 2 will be due on WEDNESDAY of week 3. Starting with assignment 3, the due dates will be on TUESDAYS!