Advanced Econometrics I
EMET4314/8014
Semester 1, 2025

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Assignment 1
(due by: Tuesday week 2, 11:00am)

Submission Instructions

Online submission per Wattle file upload only. Only pdf-files accepted. Your solution must be in your own handwriting. You can write on paper and use a suitable app on your phone to scan and upload a pdf-file, or you can use a stylus-type pen on your tablet-type device. List everyone you worked with.

Exercises

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

1. Consider the space $Z = (0, 1]$ equipped with the metric $d(x, y) = |x - y|$. Consider the following sequence in Z: $x_n = 1/n, n = 1, 2, \ldots$. Is it a Cauchy sequence? Does it converge?

2. Let X, Y be elements from a Hilbert space. Prove:

   (i) Cauchy-Schwarz inequality: $|\langle X, Y\rangle| \leq \|X\| \cdot \|Y\|$

   (ii) Triangle inequality: $\|X + Y\| \leq \|X\| + \|Y\|$

3. Prove: If $\text{E}\left(X^2\right) < \infty$ and $\text{E}\left(Y^2\right) < \infty$, then

   - $|\text{E}X| < \infty$ and $|\text{E}Y| < \infty$;
   - $|\text{E}(XY)| < \infty$;
   - $|\text{Cov}(X, Y)| < \infty$.

   This is useful: to guarantee existence of covariances, we only need finite second moments. That is why we define L_2 to be the space of random variables with finite second moments.

   Related useful fact (for your enjoyment, no need to prove):
   $\text{E}\left(|Y|^p\right) < \infty$ implies $\text{E}\left(|Y|^q\right) < \infty$ for $1 \leq q \leq p$
   (by Liapunov's inequality).

4. Prove: $\text{Cov}(X, Y) = \text{Cov}(X, \text{E}(Y|X))$

5. Prove: if $X \in \{0,1\}$ then $\frac{\text{Cov}(X,Y)}{\text{Var}X} = \text{E}(Y|X=1) - \text{E}(Y|X=0)$.

6. Consider the space L_2, as defined in the lecture. Let $X, Y \in L_2$. Prove that $\text{E}(XY)$ is an inner product.

7. Let $X_2, X_3, Y \in L_2$. Find the projection of Y on $\text{sp}\,(X_2, X_3)$. (What I'm trying to say here is that you are NOT including the constant in the span.)

   Use the following Gram-Schmidt orthogonalization procedure to construct an orthonormal set:

   Lemma 1 (Gram-Schmidt). *Let $V_1, V_2, V_3, \ldots$ be a linearly independent sequence in an inner product space. Set $U_1 = V_1/||V_1||$, and define recursively:*

$$U_k = \frac{V_k - \sum_{i=1}^{k-1} \langle V_k, U_i \rangle U_i}{\left\| V_k - \sum_{i=1}^{k-1} \langle V_k, U_i \rangle U_i \right\|}, \qquad \textit{for } k = 2, 3, \ldots$$

   Then $U_1, U_2, U_3, \ldots$ is an orthonormal sequence with sp$(U_1, U_2, \ldots, U_k) =$ sp$(V_1, V_2, \ldots, V_k)$.

8. Let $X_1, \ldots, X_K, Y \in L_2$. Use calculus to derive the following:

$$\tilde{\beta} := \underset{b \in \mathbb{R}^K}{\text{argmin}}\, \text{E}\left((Y - X'b)^2 \right),$$

   where $X := (X_1, \ldots, X_K)'$ so that dim $X =$ dim $b = K \times 1$.

   This demonstrates that $\mathbb{P}_{\text{sp}(X_1,\ldots,X_K)} Y$ can also be obtained by "traditional" methods.

This assignment will be discussed in the week 2 tutorial.