Advanced Econometrics I
EMET4314/8014
Semester 1, 2026

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Problem Set 4
(due: Monday week 5, 11:00am)

Submission Instructions: Same as last week.

Exercises

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

1. Let X be a binary random variable with $\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2}$. Let Y satisfy $Y \mid X = 0 \sim \mathcal{N}(0, 1)$, and $Y \mid X = 1 \sim \mathcal{N}(2, 4)$.
   (a) Compute $\mathrm{E}(Y \mid X)$.
   (b) Compute $\mathrm{Var}(Y \mid X)$. Is this a homoskedastic setting?
   (c) Compute $\mathrm{E}\left(\mathrm{Var}(Y \mid X)\right)$.
   (d) Compute $\mathrm{Var}\left(\mathrm{E}(Y \mid X)\right)$.
   (e) Using parts (c) and (d), compute $\mathrm{Var}(Y)$ from the law of total variance:
   $$\mathrm{Var}(Y) = \mathrm{E}\left(\mathrm{Var}(Y \mid X)\right) + \mathrm{Var}\left(\mathrm{E}(Y \mid X)\right).$$
2. In the linear regression model in which X_{i1} is the constant term, show that we can safely presume that $\mathrm{E}(e_i) = 0$. (This exercise justifies why we always want to include a constant term when running regressions.)
3. Prove that the OLS estimator $\hat{\beta}^{\mathrm{OLS}}$ for β in the linear regression model is consistent. In your derivation, employ the $\mathrm{o}_p(1)$ and $\mathrm{O}_p(1)$ notation!
4. The linear regression model in matrix format is $Y = X\beta + e$, with the usual definitions. Let $\mathrm{E}(e|X) = 0$ and
   $$\mathrm{E}(ee'|X) = \sigma^2\Gamma = \sigma^2 \begin{bmatrix} \gamma_1 & 0 & 0 & \cdots & 0 \\ 0 & \gamma_2 & 0 & \cdots & 0 \\ & & & & \\ \vdots & & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \gamma_N \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & 0 & \cdots & 0 \\ & & & & \\ \vdots & & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \sigma_N^2 \end{bmatrix} = \Sigma.$$
   Notice that as a covariance matrix, Σ is symmetric and nonnegative definite.

(a) Derive Var$(\hat{\beta}^{\text{OLS}}|X)$.

(b) Let $\tilde{\beta} := C'Y$ be any other linear unbiased estimator where C is an $N \times K$-dimensional matrix that is based on X. Following the lecture notes, prove that Var$(\tilde{\beta}|X) \geq (X'\Sigma^{-1}X)^{-1}$.

(c) An oracle tells you Γ. Let $\tilde{Y} := \Gamma^{-1/2}Y$ and $\tilde{X} := \Gamma^{-1/2}X$. Define the ***generalized least squares (GLS)*** estimator $\hat{\beta}_{GLS} := (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{Y}$. This defines the GLS estimator as the OLS estimator of $\tilde{Y}$ on $\tilde{X}$.

    Derive Var$(\hat{\beta}_{GLS}|X)$. How does it compare to Var$(\tilde{\beta}|X)$ from part (b)?