THE AUSTRALIAN NATIONAL UNIVERSITY

First Semester Final Examination– June, 2020

Advanced Econometrics I

(EMET 4314/8014)

Reading Time: 0 Minutes
Writing Time: 120 Minutes

Instructions

- Answer all 4 questions of this handout.
- Write on paper or an electronic device (such as an Ipad).
- Handwritten answers only! Do not type anything.
- Work on your answers only between 3:00pm–5:00pm!
- Upload your file at 5:00pm sharp!
- If written on a piece of paper, scan your work and upload to Wattle.
- If using an electronic device, upload your file to Wattle.
- Provide complete, self-contained, and correct answers!
- Make reasonable assumptions where necessary.
- Justify all steps that are not obvious.
- Good luck!

Beginning of Exam Questions

1. [1 mark total] Write the following statement by hand:

   I hereby declare

   - to uphold the principles of academic integrity, as defined in the University Academic Misconduct Rules;
   - that your work in the final exam in no part involves copying, cheating, collusion, fabrication, plagiarism or recycling.

Important
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2. [20 marks total]

   Consider the conditional expectation function $\mathrm{E}(Y_i|X_i) = g(X_i, \beta)$ where

   - X_i is a scalar
   - Y_i is binary: $Y_i \in \{0, 1\}$

   (a) [10 marks] Assume $g(X_i, \beta) = X_i\beta$. Propose an estimator for β that is unbiased, consistent, and efficient. Prove that it is unbiased and consistent. Derive its asymptotic distribution.

   (b) Assume $g(X_i, \beta) = F(X_i\beta)$ for some differentiable function F that is bounded between zero and one.

      i. [2 marks] Show that $Y_i = F(X_i\beta) + e_i$ such that $\mathrm{E}(e_i|X_i) = 0$.

      ii. [2 marks] Conditional on X_i, the error e_i can take on two values

$$e_i = \begin{cases} 1 - F(X_i\beta) & \text{if } Y_i = 1 \\ -F(X_i\beta) & \text{if } Y_i = 0 \end{cases}$$

      Show that $\mathrm{Var}\,(e_i|X_i) = F(X_i\beta) \cdot (1 - F(X_i\beta))$.
      (Notice that $\Pr(Y_i = 1|X_i) = \mathrm{E}(Y_i|X_i) = F(X_i\beta)$.)

      iii. [4 marks] Consider the following least squares estimator:

$$\hat{\beta} := \operatorname*{argmin}_b \sum_{i=1}^{N} \frac{\Big(Y_i - F(X_i b)\Big)^2}{\omega_i},$$

      where $\omega_i := F(X_i\beta) \cdot (1 - F(X_i\beta))$.
      Why are we dividing by ω_i in the objective function? Is $\hat{\beta}$ a feasible estimator? If not, suggest a feasible estimator.

      iv. [2 marks] Derive the first order condition for the minimization problem in part iii. Does the first order condition look familiar?

3. `[20 marks total - 5 marks each]`

Are the following statements true or false? Provide a complete yet short explanation. Use mathematical derivations where necessary.

(Note: you will not receive any credit without providing a correct explanation.)

(a) If $\mathrm{E}\left(X^2\right) < \infty$ and $\mathrm{E}\left(Y^2\right) < \infty$, then $|\mathrm{E}(XY)| < \infty$.

(b) Let $Y_1, \ldots, Y_N$ be independent random variables, each with pdf

$$f(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y-\mu)^2\right).$$

The maximizer of the joint log-likelihood function is equal to $\bar{Y}$.

(c) In the model $Y_i = X_i\beta + e_i$ where X_i is a scalar and $\mathrm{E}(e_i X_i) = 0$ you want to test the null hypothesis $\beta = 0$. You suggest the following hypothesis test: draw a number randomly from $\{1, 2, \ldots, 10\}$ and reject the null if that number equals 4.

That hypothesis test has a size of 5%.

(d) Let Y_i be independent random variables (with $i = 1, 2, \ldots$), each with pdf

$$f(y) = \begin{cases} \frac{1}{c} & \text{for } y \in (0, c) \\ 0 & \text{elsewhere} \end{cases}$$

for some $0 < c < \infty$. Then $Y_1, Y_2, \ldots$ is bounded in probability.

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4. `[20 marks total]`

Consider the moment conditions $\mathrm{E}\left(\psi(W_i, \theta_0)\right) = 0$, where W_i represents the variables of your model. The function $\psi(W_i, \cdot)$ is twice continuously differentiable. Let $\dim \psi(W_i, \theta_0) = \dim \theta_0 = K \times 1$.

An obvious analog estimator for θ_0 is $\hat{\theta}$ which solves $(1/N)\sum_{i=1}^{N} \psi(W_i, \hat{\theta}) = 0$.

You may assume throughout that $\hat{\theta} = \theta_0 + o_p(1)$.

To establish the asymptotic distribution of $\hat{\theta}$ you will use a mean value theorem approximation, like so:

$$0 = (1/N)\sum_{i=1}^{N} \psi(W_i, \theta_0) + \left(\frac{1}{N}\sum_{i=1}^{N} \frac{\partial \psi}{\partial \theta'}(W_i, \tilde{\theta})\right)\left(\hat{\theta} - \theta_0\right).$$

(a) `[2 marks]` State the approximate distribution of $(1/\sqrt{N})\sum_{i=1}^{N} \psi(W_i, \theta_0)$? (This should be brief.)

(b) `[2 marks]` What is the probability limit of $\displaystyle \frac{1}{N}\sum_{i=1}^{N} \frac{\partial \psi}{\partial \theta'}(W_i, \tilde{\theta})$?

(c) `[2 marks]` The above mean value approximation can be used to obtain the following asymptotic distribution:

$$\sqrt{N}(\hat{\theta} - \theta_0) \xrightarrow{\mathrm{d}} \mathrm{N}(0, PQP').$$

Determine P and Q here.

(Note: I am NOT asking you to derive the asymptotic distribution.)

`Note: This exercise continues on next page …`

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Now consider the model

$$\begin{aligned} Y_i &= \beta_0 X_i + e_i \\ X_i &= Z_i' \pi_0 + v_i, \end{aligned}$$

where $\dim(X_i) = 1 \times 1$ and $\dim Z_i = L \times 1$ with $L > 1$. Let

$$\begin{pmatrix} e_i \\ v_i \end{pmatrix} \Bigg| Z_i \sim \mathrm{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_e^2 & \sigma_{ev} \\ \sigma_{ev} & \sigma_v^2 \end{pmatrix} \right).$$

Let

$$\psi(W_i, \beta, \pi) := \begin{pmatrix} \psi_1(W_i, \pi) \\ \psi_2(W_i, \pi, \beta) \end{pmatrix},$$

where

$$\begin{aligned} \psi_1(W_i, \pi) &:= Z_i(X_i - Z_i'\pi) \\ \psi_2(W_i, \beta, \pi) &= (Z_i'\pi) \cdot (Y_i - \beta(Z_i'\pi)). \end{aligned}$$

For brevity we may also write $\psi(\beta, \pi)$ instead of $\psi(W_i, \beta, \pi)$.

(d) `[2 marks]` Let $u_i := Y_i - \beta_0(Z_i'\pi_0)$. Derive the distribution of $\begin{pmatrix} u_i \\ v_i \end{pmatrix} \Bigg| Z_i$.

(e) `[1 marks]` Let $\theta := (\pi', \beta)'$. Derive $\frac{\partial \psi}{\partial \theta'}(\beta_0, \pi_0)$.

(f) `[3 marks]` Using $\Gamma := \mathrm{E}(Z_i Z_i')$, determine $\mathrm{E}\left(\frac{\partial \psi}{\partial \theta'}(\beta_0, \pi_0) \right)$.

(g) `[2 marks]` Determine $\left(\mathrm{E}\left(\frac{\partial \psi}{\partial \theta'}(\beta_0, \pi_0) \right) \right)^{-1}$.

Hint: make use of

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}.$$

(h) `[3 marks]` Determine $\mathrm{E}\Big(\psi(\beta_0, \pi_0) \cdot \psi(\beta_0, \pi_0)' \Big)$.

(i) `[3 marks]` Let $\hat{\beta}$ be the obvious analog estimator of β_0. What is the asymptotic distribution of $\sqrt{N}\,(\hat{\beta} - \beta_0)$? Be specific about its asymptotic variance! (Do not provide the asymptotic variance of $\hat{\pi}$.)

`End of Exam`