THE AUSTRALIAN NATIONAL UNIVERSITY

First Semester Final Examination– June, 2021

Advanced Econometrics I

(EMET 4314/8014)

Reading Time: 0 Minutes
Writing Time: 120 Minutes

Instructions

- Answer all 4 questions of this handout.
- Write on paper or an electronic device (such as an Ipad).
- Handwritten answers only! Do not type anything.
- Work on your answers only between 9:30am–11:30am.
- If written on a piece of paper, scan your work and upload to Wattle.
- If using an electronic device, upload your file to Wattle.
- Upload your file as soon as possible after 11:30am.
  You have at most 15 minutes to scan and upload your document.
- Provide complete, self-contained, and correct answers.
- Make reasonable assumptions where necessary.
- Justify all steps that are not obvious.

Beginning of Exam Questions

1. `[1 mark total]` Write the following statement by hand:

   I hereby declare

   - to uphold the principles of academic integrity, as defined in the University Academic Misconduct Rules;
   - that your work in the final exam in no part involves copying, cheating, collusion, fabrication, plagiarism or recycling.

Important
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2. `[20 marks total]`

   Consider the conditional expectation function $\mathrm{E}(Y_i|X_i) = g(X_i, \beta)$ where

   - $\dim X_i = \dim \beta = K \times 1$
   - Y_i is binary: $Y_i \in \{0, 1\}$

   (a) `[8 marks]`
   Assume $g(X_i, \beta) = X_i'\beta$. Prove that the OLS estimator for β is unbiased and consistent. Derive its asymptotic distribution.
   (Use stochastic order notation, such as "$o_p(1)$", as much as possible. Provide detailed and comprehensive derivations!)

   (b) Assume $g(X_i, \beta) = \Phi(X_i'\beta)$, where Φ is the cdf of the standard normal distribution.

   (i) `[2 marks]` Show that $Y_i = \Phi(X_i'\beta) + e_i$ such that $\mathrm{E}(e_i|X_i) = 0$.

   (ii) `[2 marks]` Show that Var $(e_i|X_i) = \Phi(X_i'\beta) \cdot (1 - \Phi(X_i'\beta))$.

   (iii) `[2 marks]` Consider the following least squares estimator:

   $$\hat{\beta} := \underset{b}{\operatorname{argmin}} \sum_{i=1}^{N} \frac{\left(Y_i - \Phi(X_i'b)\right)^2}{\omega_i},$$

   where $\omega_i := \Phi(X_i'\beta) \cdot (1 - \Phi(X_i'\beta))$.
   The estimator $\hat{\beta}$ is not feasible. Suggest a feasible estimator.

   (iv) `[2 marks]` Derive the first order condition for the minimization problem in part (iii).

   (c) `[4 marks]`
   Discuss briefly: What are the advantages of using $g(X_i, \beta) = \Phi(X_i'\beta)$ instead of $g(X_i, \beta) = X_i'\beta$?

3. `[25 marks total]`

Are the following statements true or false? Provide a complete explanation. Use mathematical derivations where necessary.

(Note: you will not receive any credit without providing a correct explanation.)

(a) `[2 marks]`

Let $Y_1, \ldots, Y_{200}$ be a random sample with $\mathrm{E}(Y_i) = \mu$ and $0 < \mathrm{Var}\, Y_i < \infty$. Define $\bar{Y} := \sum_{i=1}^{200} Y_i/200$. Then $(Y_2 + Y_4 + \cdots + Y_{200})/100$ is an unbiased estimator for μ with a variance greater than that of $\bar{Y}$.

(b) `[5 marks]`

Let $Y_1, \ldots, Y_N$ be independent random variables, each with pdf

$$f(y) = (1-\alpha)^{1-y}\alpha^y, \qquad \text{for } y \in \{0,1\}, \qquad 0 \le \alpha \le 1,$$

and zero elsewhere. The maximum likelihood estimator of α is equal to the sample average $\bar{Y}$.

(c) `[8 marks]`

Consider the model $Y_i = X_i\beta + e_i$ where X_i is a scalar, $\mathrm{E}(e_iX_i) = 0$, and the error is homoskedastic. Let $\hat{\beta}$ be the OLS estimator of β. You want to test the null hypothesis $\beta = 0$. You suggest the following hypothesis test: Reject the null whenever the absolute value of

$$\sqrt{N}\hat{\beta} \cdot \frac{\sqrt{\sum_{i=1}^N X_i^2}}{\sqrt{\sum_{i=1}^N (Y_i - X_i\hat{\beta})^2}}$$

exceeds 1.96. That hypothesis test has a size of 5%.

(d) `[10 marks]`

Let $Y_1, Y_2, \ldots$ be random variables with $\mathrm{E}(Y_i) = \mu$ and $\mathrm{Var}\,(Y_i) = \sigma_i^2 < \infty$ for all i. Let $\mathrm{Cov}(Y_s, Y_t) = 0$ whenever $s \neq t$. Then $\sum_{i=1}^N Y_i/N = \mu + o_p(1)$.

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4. `[25 marks total]`

Consider the moment conditions $\mathrm{E}\left(\psi(W_i, \theta_0)\right) = 0$, where W_i represents the variables of your model. The function $\psi(W_i, \cdot)$ is twice continuously differentiable. Let $\dim \psi(W_i, \theta_0) = \dim \theta_0 = K \times 1$.

An obvious analog estimator for θ_0 is $\hat{\theta}$ which solves $(1/N) \sum_{i=1}^{N} \psi(W_i, \hat{\theta}) = 0$.

You may assume throughout that $\hat{\theta} = \theta_0 + o_p(1)$.

To establish the asymptotic distribution of $\hat{\theta}$ you will use a mean value theorem approximation, like so:

$$0 = (1/N) \sum_{i=1}^{N} \psi(W_i, \theta_0) + \left(\frac{1}{N} \sum_{i=1}^{N} \frac{\partial \psi}{\partial \theta'}(W_i, \tilde{\theta}) \right) \left(\hat{\theta} - \theta_0\right).$$

(a) `[2 marks]` State the approximate distribution of $(1/\sqrt{N}) \sum_{i=1}^{N} \psi(W_i, \theta_0)$. (Brief.)

(b) `[2 marks]` What is the probability limit of $\frac{1}{N} \sum_{i=1}^{N} \frac{\partial \psi}{\partial \theta'}(W_i, \tilde{\theta})$?

(c) `[2 marks]` The above mean value approximation can be used to get this asymptotic distribution: $\sqrt{N}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathrm{N}(0, PQP')$. Determine P and Q.

For the model $Y_i = X_i \theta_0 + e_i$, with X_i a scalar, and $\mathrm{E}(e_i | X_i) = 0$, consider the estimator

$$\theta^* := \operatorname*{argmin}_{t \in \mathbb{R}} \left\{ \left(\sum_{i=1}^{N} (Y_i - X_i t)^2 \right) + c \cdot t^2 \right\},$$

where c is a known positive constant.

(d) `[3 marks]` Determine the function ψ for the estimator θ^*.

(e) `[3 marks]` Solve for θ^*.

(f) `[4 marks]` Is θ^* unbiased?

(g) `[2 marks]` Derive $\frac{\partial \psi}{\partial \theta'}(W_i, \theta_0)$.

(h) `[2 marks]` Determine $\mathrm{E}\left(\frac{\partial \psi}{\partial \theta'}(W_i, \theta_0) \right)$.

(i) `[2 marks]` Determine $\mathrm{E}\Big(\psi(W_i, \theta_0) \cdot \psi(W_i, \theta_0)' \Big)$.

(j) `[3 marks]` Derive the asymptotic distribution and variance of $\sqrt{N}\,(\theta^* - \theta_0)$.

`End of Exam`