Advanced Econometrics I EMET4314/8014 Semester 1, 2025 Juergen Meinecke Research School of Economics ANU

Assignment 5

(due: Tuesday week 6, 11:00am)

Submission Instructions: Same as last week.

Exercises

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

1. Consider the scalar model $Y_i = \beta_1 X_{i1} + \beta_2 X_{i2} + e_i$ where both variables are endogenous, that is, $E(e_i X_{i1}) \neq 0$ and $E(e_i X_{i2}) \neq 0$. You have available one instrumental variable Z_i (also a scalar) such that $E(e_i Z_i) = 0$ and $E(X_{i1} Z_i) \neq 0$ and $E(X_{i2} Z_i) \neq 0$.

Consider the two reduced form regressions

$$X_{i1} = \pi Z_i + v_i$$
$$X_{i2} = \gamma Z_i + w_i,$$

where $E(v_i X_{i1}) = 0$ and $E(w_i X_{i2}) = 0$.

Show that a regression of Y_i on Z_i cannot separately identify β_1 and β_2 , but instead only $\pi\beta_1 + \gamma\beta_2$.

Contrast this with the case in which you have two instruments:

$$X_{i1} = \pi_1 Z_{i1} + \pi_2 Z_{i2} + v_i$$

$$X_{i2} = \gamma_1 Z_{i1} + \gamma_2 Z_{i2} + w_i.$$
(1)

Plugging into the structural equation results in

$$Y_{i} = (\beta_{1}\pi_{1} + \beta_{2}\gamma_{1})Z_{i1} + (\beta_{1}\pi_{2} + \beta_{2}\gamma_{2})Z_{i2} + (e_{i} + \beta_{1}v_{i} + \beta_{2}w_{i}).$$
(2)

OLS estimation of the two reduced form regressions (1) will produce consistent estimates for π_1, π_2, γ_1 and γ_2 . Likewise, OLS estimation of the transformed structural equation (2) will produce consistent estimates for $(\beta_1\pi_1 + \beta_2\gamma_1)$ and $(\beta_1\pi_2 + \beta_2\gamma_2)$. You are now able to back out estimates for β_1 and β_2 , because you essentially are facing a problem of solving two equations for two unknowns. (When you only have one instrumental variable, then you are solving one equation for two unknowns.)

Note: this exercise goes back to a question that was raised during the week 5 lecture. It illustrates why the number of instrumental variables must be at least as large as the number of endogenous regressors.

2. Remember the following parameter from the week 5 lecture:

 $\pi = \mathbf{E}(Z_i Z_i')^{-1} \mathbf{E}(Z_i X_i'),$

where $X_i = (X'_{i1}, X'_{i2})'$ and $Z_i = (X'_{i1}, Z'_{i2})'$. I claimed that π is an upper triangular block matrix. Show this!

Make use of the following result for the inverse of *partitioned* (or block) matrices (taken from Greene's textbook):

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} A_{11}^{-1} (I + A_{12}F_2A_{21}A_{11}^{-1}) & -A_{11}^{-1}A_{12}F_2 \\ -F_2A_{21}A_{11}^{-1} & F_2 \end{pmatrix},$$

where $F_2 := (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$.

3. Let $Y_i = X_i\beta + e_i$ where X_i is a scalar random variable and $E(e_i|X_i) = 0$. You observe (\widetilde{X}_i, Y_i) with $\widetilde{X}_i := X_i + r_i$ where r_i is a random error. Derive the probability limit of the OLS estimator in the regression of Y_i on \widetilde{X}_i . For simplicity, assume that $EX_i = Er_i = 0$.

What is the probability limit equal to when the variance of r_i is zero? When the variance of r_i is very large? What is the intuition here?

Hint: Your probability limit should have the form $\beta(1 - stuff)$, where stuff depends only on the population variances of r_i and X_i .

Note: This phenomenon is usually referred to as *measurement error bias* or *attenuation bias*. It is not good to use the word *bias* here though. After all, we are not studying the expected value of the OLS estimator, instead we are studying its probability limit and show that it does not equal β . Biasedness and inconsistency are not the same thing.

- 4. Prove that $\hat{\beta}^{\text{IV}}$ is consistent.
- 5. Derive the asymptotic distribution of $\sqrt{N}(\hat{\beta}^{IV} \beta)$.