Advanced Econometrics I
EMET4314/8014
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Assignment 2
(due: Wednesday week 3, 11:00am)

Submission Instructions: Same as last week.

The solutions will be discussed in the Friday workshop during week 3. Please let me know which exercises I should focus on.

Exercises

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

1. Let $X_2, X_3, Y \in L_2$. Use calculus to derive the following:

$$\widetilde{\beta}_2, \widetilde{\beta}_3 := \underset{b_2, b_3}{\operatorname{argmin}} \operatorname{E}\left((Y - X_2 b_2 - X_3 b_3)^2\right).$$

   Provide explicit and fully derived solutions for $\widetilde{\beta}_2$ and $\widetilde{\beta}_3$! Do not use linear algebra!

   Compare your results to the projections of Y on $\operatorname{sp}(X_2, X_3)$ (from assignment 1) and of Y on $\operatorname{sp}(1, X_2, X_3)$ (from the lecture).

2. Let $Y_i \in L_2$ for $i = 1, \ldots, N$ be a scalar random variables with independent and identical distribution with $\mu_Y := \operatorname{E} Y_i$ and $\sigma_Y^2 := \operatorname{Var} Y_i < \infty$. The sample average is defined as $\bar{Y}_N := \sum_{i=1}^N Y_i / N$. Derive $\operatorname{E}\bar{Y}_N$ and $\operatorname{Var}\bar{Y}_N$.

3. Using the same definitions as in exercise (2), define

$$Z_N := \frac{\bar{Y}_N - \operatorname{E}\bar{Y}_N}{\sqrt{\operatorname{Var}\bar{Y}_N}}.$$

   Derive $\operatorname{Var} Z_N$.

   This result illustrates that, if the limit distribution of Z_N exists, it will be non-degenerate. That is, it does not just collapse to a point.

   Remark:
   You may correctly conjecture, based on the CLT, that the limit distribution is standard normal under mild conditions. Proving this requires some not too difficult manipulations of moment generating (or characteristic) functions.

4. The definition of the OLS estimator using matrix notation is:

$$\hat{\beta}^{\text{OLS}} := \underset{b \in \mathbb{R}^K}{\text{argmin}} (Y - Xb)'(Y - Xb),$$

   where $\dim X = N \times K$ and $\dim Y = N \times 1$. Derive $\hat{\beta}^{\text{OLS}}$ using calculus.
   The following tools from matrix calculus may be helpful:

   Lemma 1.

$$\frac{\partial Az}{\partial z} = A' \qquad \qquad \frac{\partial (z'Az)}{\partial z} = (A + A')z$$

5. Let $Y_i = X_i'\beta^* + u_i$ where $\text{E}(X_i u_i) = 0$, where $\dim X_i = K \times 1$ and $\dim Y_i = 1 \times 1$. Define

$$\hat{\theta} := \left(\left(\sum_{i=1}^N X_i X_i' \right) + \lambda I_K \right)^{-1} \left(\sum_{i=1}^N X_i Y_i \right),$$

   where $\lambda > 0$ and I_K is the K-dimensional identity matrix. Derive the probability limit of $\hat{\theta}$. In your derivation, make use of the $\text{o}_p(1)$ and $\text{O}_p(1)$ notation. Is $\hat{\theta}$ consistent for β^*?