Advanced Econometrics I EMET4314/8014 Semester 1, 2025 Juergen Meinecke Research School of Economics ANU

## **Assignment 3**

(due: Tuesday week 4, 11:00am)

Submission Instructions: Same as last week.

## Exercises

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

- 1. Let Z be a random variable with  $EZ^2 < \infty$ . Prove that  $Z_N \xrightarrow{d} Z$  implies  $Z_N = O_p(1)$ .
- 2. The pdf of a normal distribution is  $f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right)$ , for  $-\infty < y < \infty$ .
  - (a) Derive the moment generating function of a normally distributed random variable. Denote it by  $M_Y(t; (\mu, \sigma))$ .
  - (b) Take the first two derivatives of  $M_Y(t; (\mu, \sigma))$  and evaluate them at zero.
  - (c) Evaluate the mgf for the standard normal case:  $M_Y(t; (0, 1))$ . (This proves a Lemma from the week 3 lecture notes.)
- 3. Let  $Y = X\beta^* + u$  with dim  $X = N \times K$  and the usual definitions. Define the *projection* matrix  $P_X := X(X'X)^{-1}X'$  and the *residual maker matrix*  $M_X := I_N P_X$ . Show that:
  - (i)  $P_X Y = \hat{Y}$  (hence the name *projection matrix*)
  - (ii)  $M_X Y = \hat{u}$  (hence the name *residual maker matrix*)
  - (iii)  $M_X u = \hat{u}$
  - (iv) Symmetry:  $P_X = P'_X$  and  $M_X = M'_X$
  - (v) Idempotency:  $P_X P_X = P_X$  and  $M_X M_X = M_X$
  - (vi) tr  $P_X = K$  and tr  $M_X = N K$
- 4. Use a derivation similar to lecture notes 3 to show that  $\sum_{i=1}^{N} \hat{u}_i^2 / (N K)$  is an unbiased estimator for  $\sigma_u^2$ .

- 5. Consider the asymptotic distribution of  $\sqrt{N}(\hat{\beta}^{OLS} \beta^*)$  under the assumption of *homoskedasticity*, that is:  $E(u_i^2|X_i) = \sigma_u^2$  where  $\sigma_u^2 \in \mathbb{R}$ . Note, as usual,  $\beta^* = E(X_iX_i')^{-1}E(X_iY_i)$ .
  - (a) Derive the asymptotic distribution of  $\sqrt{N}(\hat{\beta}^{\text{OLS}} \beta^*)$  under homoskedasticity. Justify each step!
  - (b) Suggest a consistent estimator for the asymptotic variance of  $\sqrt{N}(\hat{\beta}^{\text{OLS}} \beta^*)$  under homoskedasticity.
  - (c) Prove that your estimator from part (b) is consistent. In your proof, make use of the  $o_p(1)$  and  $O_p(1)$  notation. Justify each step!