Advanced Econometrics I
EMET4314/8014
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Assignment 3
(due: Tuesday week 4, 11:00am)

Submission Instructions: Same as last week.

The solutions will be discussed in the Friday workshop during week 4. Please let me know which exercises I should focus on.

Exercises

Provide transparent derivations. Justify steps that are not obvious. Use self sufficient proofs. Make reasonable assumptions where necessary.

1. Let Z be a random variable with $\mathrm{E}Z^2 < \infty$. Prove that $Z_N \xrightarrow{\mathrm{d}} Z$ implies $Z_N = \mathrm{O}_p(1)$.
2. Let $Y = X\beta^* + u$ with $\dim X = N \times K$ and the usual definitions. Define the *projection matrix* $P_X := X(X'X)^{-1}X'$ and the *residual maker matrix* $M_X := I_N - P_X$. Show that:
   (i) $P_X Y = \hat{Y}$ (hence the name *projection matrix*)
   (ii) $M_X Y = \hat{u}$ (hence the name *residual maker matrix*)
   (iii) $M_X u = \hat{u}$
   (iv) Symmetry: $P_X = P_X'$ and $M_X = M_X'$
   (v) Idempotency: $P_X P_X = P_X$ and $M_X M_X = M_X$
   (vi) tr $P_X =$ rank $P_X = K$ and tr $M_X =$ rank $M_X = N - K$
   Hint: Use the spectral decomposition for symmetric matrices: $A = C\Lambda C'$ where Λ is the diagonal matrix collecting all real eigenvalues on the diagonal and C is an eigenvector matrix satisfying $C'C = I$.
3. Use a derivation similar to lecture notes 3 to show that $\sum_{i=1}^N \hat{u}_i^2/(N-K)$ is an unbiased estimator for σ_u^2.
4. Consider the asymptotic distribution of $\sqrt{N}(\hat{\beta}^{\mathrm{OLS}} - \beta^*)$ under the assumption of *homoskedasticity*, that is: $\mathrm{E}(u_i^2|X_i) = \sigma_u^2$ where $\sigma_u^2 \in \mathbb{R}$. Note, as usual, $\beta^* = \mathrm{E}(X_i X_i')^{-1}\mathrm{E}(X_i Y_i)$.
   (a) Derive the asymptotic distribution of $\sqrt{N}(\hat{\beta}^{\mathrm{OLS}} - \beta^*)$ under homoskedasticity. Justify each step!
   (b) Suggest a consistent estimator for the asymptotic variance of $\sqrt{N}(\hat{\beta}^{\mathrm{OLS}} - \beta^*)$ under homoskedasticity.
   (c) Prove that your estimator from part (b) is consistent. In your proof, make use of the $\mathrm{o}_p(1)$ and $\mathrm{O}_p(1)$ notation. Justify each step!